数学演習第一（演習第９回）　【解答例】

微積：漸近展開, 積分の計算 (1)　　2020 年 7 月 22 日

$\boxed{0}$ 並んでいる順に $\boxed{\dfrac{h(x)}{x^n}}$, $\boxed{\dfrac{1}{n!}}$, $\boxed{\dfrac{(-1)^n}{(2n)!}}$, $\boxed{\dfrac{(-1)^n}{(2n+1)!}}$, $\boxed{\dfrac{(-1)^{n-1}}{n}}$, $\boxed{\dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}}$, $\boxed{(-1)^n}$.

$\boxed{1}$ (本問でランダウの記号 $o(x^n)$ を使うときはいつも，$x \to 0$ を省略している.)

(1) (d) $(\alpha=-\frac{1}{2})$ を用いて， $\dfrac{1}{\sqrt{1+x}} = 1 + \dfrac{-\frac{1}{2}}{1!}x + \dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + o(x^2) = \boxed{1-\dfrac{1}{2}x+\dfrac{3}{8}x^2+o(x^2)}$.

(2) (d) $(\alpha=\frac{1}{2})$ を用いて，$\sqrt{1+x} = 1 + \dfrac{\frac{1}{2}}{1!}x + \dfrac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + o(x^2) = \boxed{1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+o(x^2)}$.

(3) (a) より $e^{\pm x} = 1 \pm x + \dfrac{1}{2}x^2 \pm \dfrac{1}{6}x^3 + \dfrac{1}{24}x^4 \pm \dfrac{1}{120}x^5 + o(x^5)$ (複号同順) であるから,

$$\cosh x = \frac{e^x+e^{-x}}{2} = \boxed{1+\frac{x^2}{2}+\frac{x^4}{24}+o(x^5)},\quad \sinh x = \frac{e^x-e^{-x}}{2} = \boxed{x+\frac{x^3}{6}+\frac{x^5}{120}+o(x^5)}.$$

(4) (a) を用いて，$2^x = (e^{\log 2})^x = e^{(\log 2)x} = \boxed{1+(\log 2)x+\dfrac{(\log 2)^2}{2}x^2+o(x^2)}$.

(5) (c) より, $\log(3+x) = \log 3 + \log\Big(1+\dfrac{x}{3}\Big) = \log 3 + \dfrac{x}{3} - \dfrac{1}{2}\Big(\dfrac{x}{3}\Big)^2 + o(x^2) = \boxed{\log 3+\dfrac{1}{3}x-\dfrac{1}{18}x^2+o(x^2)}$.

(6) 部分分数分解した後に (d) $(\alpha=-1)$ を用いて,

$$\begin{aligned}\frac{x}{2+x-x^2} &= \frac{x}{(2-x)(1+x)} = \frac{1}{3}\Big(\frac{2}{2-x}-\frac{1}{1+x}\Big) = \frac{1}{3}\Big(\frac{1}{1-\frac{x}{2}}-\frac{1}{1+x}\Big)\\ &= \frac{1}{3}\Big\{\Big(1+\frac{x}{2}+\Big(\frac{x}{2}\Big)^2+\Big(\frac{x}{2}\Big)^3+o(x^3)\Big)-\Big(1-x+x^2-x^3+o(x^3)\Big)\Big\} = \boxed{\frac{1}{2}x-\frac{1}{4}x^2+\frac{3}{8}x^3+o(x^3)}.\end{aligned}$$

【別法】 $\dfrac{x}{2+x-x^2} = \dfrac{x}{2}\cdot\dfrac{1}{1-\frac{x}{2}}\cdot\dfrac{1}{1+x} = \dfrac{x}{2}\Big(1+\dfrac{x}{2}+\Big(\dfrac{x}{2}\Big)^2+o(x^2)\Big)\Big(1-x+x^2+o(x^2)\Big) = \dfrac{x}{2}-\dfrac{x^2}{4}+\dfrac{3x^3}{8}+o(x^3)$.

(7) (c) より，$\log(1\pm x) = \pm x - \dfrac{x^2}{2} \pm \dfrac{x^3}{3} - \dfrac{x^4}{4} \pm \dfrac{x^5}{5} + o(x^5)$ (複号同順) であるから,

$$\log\sqrt{\frac{1+x}{1-x}} = \frac{1}{2}\{\log(1+x)-\log(1-x)\} = \boxed{x+\frac{x^3}{3}+\frac{x^5}{5}+o(x^5)}.$$

(8) (b) より, $\cos x = 1-\dfrac{x^2}{2}+\dfrac{x^4}{24}-\dfrac{x^6}{720}+o(x^6)$. よって, $\dfrac{1-\cos x}{x^2} = \boxed{\dfrac{1}{2}-\dfrac{x^2}{24}+\dfrac{x^4}{720}+o(x^4)}$.

(9) 半角の公式と (8) の結果を用いて,

$$\Big(\frac{\sin x}{x}\Big)^2 = \frac{\sin^2 x}{x^2} = 2\cdot\frac{1-\cos 2x}{(2x)^2} = 2\Big\{\frac{1}{2}-\frac{(2x)^2}{24}+\frac{(2x)^4}{720}+o(x^4)\Big\} = \boxed{1-\frac{x^2}{3}+\frac{2x^4}{45}+o(x^4)}.$$

【別法】 $\Big(\dfrac{\sin x}{x}\Big)^2 = \Big(1-\dfrac{x^2}{6}+\dfrac{x^4}{120}+o(x^4)\Big)^2 = 1-\dfrac{x^2}{3}+\dfrac{2x^4}{45}+o(x^4)$.

(10) (a), (b) を用いて，$e^{-x}\cos x = \Big(1-x+\dfrac{x^2}{2}-\dfrac{x^3}{6}+o(x^3)\Big)\Big(1-\dfrac{x^2}{2}+o(x^3)\Big) = \boxed{1-x+\dfrac{1}{3}x^3+o(x^3)}$.

(11) (b) より, $\dfrac{x}{\sin x} = \dfrac{1}{1+(-\frac{1}{6}x^2+\frac{1}{120}x^4+o(x^4))}$. ここで, $\dfrac{1}{1+X} = 1-X+X^2+o(X^2)$ $(X\to 0)$ を用いて,

$$\frac{x}{\sin x} = 1-\Big(-\frac{1}{6}x^2+\frac{1}{120}x^4+o(x^4)\Big)+\Big(-\frac{1}{6}x^2+o(x^2)\Big)^2 = \boxed{1+\frac{1}{6}x^2+\frac{7}{360}x^4+o(x^4)}.$$

(12) $\cos x = 1+\Big(-\dfrac{x^2}{2}+\dfrac{x^4}{24}+o(x^4)\Big)$ $(x\to 0)$, $e^{1+X} = e\cdot e^X = e\Big(1+X+\dfrac{X^2}{2}+o(X^2)\Big)$ $(X\to 0)$ より,

$$\begin{aligned}e^{\cos x} &= e\Big\{1+\Big(-\frac{x^2}{2}+\frac{x^4}{24}+o(x^4)\Big)+\frac{1}{2}\Big(-\frac{x^2}{2}+o(x^2)\Big)^2+o(x^4)\Big\}\\ &= e\Big\{1+\Big(-\frac{x^2}{2}+\frac{x^4}{24}\Big)+\frac{1}{2}\cdot\frac{x^4}{4}\Big\}+o(x^4) = \boxed{e-\frac{e}{2}x^2+\frac{e}{6}x^4+o(x^4)}.\end{aligned}$$

(13) $\mathrm{Tan}^{-1}\,x = \displaystyle\int_0^x \frac{dt}{1+t^2} = \int_0^x \Big(1-t^2+t^4+o(t^4)\Big)\,dt = \boxed{x-\frac{x^3}{3}+\frac{x^5}{5}+o(x^5)}$.

(14) (1) の結果を用いて，$\mathrm{Sin}^{-1}\,x = \displaystyle\int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \Big(1-\frac{-t^2}{2}+\frac{3(-t^2)^2}{8}+o(t^4)\Big)\,dt = \boxed{x+\frac{x^3}{6}+\frac{3x^5}{40}+o(x^5)}$.

$\boxed{2}$ (1) (i) $\dfrac{1}{x^2}-\dfrac{1}{\tan^2 x} = \dfrac{\sin^2 x - x^2\cos^2 x}{x^2\sin^2 x}$ (分母 $x^2\sin^2 x$ の漸近展開が x^4 の項から始まるから, 分子を $o(x^4)$ を用いて表す)

$$\begin{aligned}&= \frac{(x-\frac{x^3}{6}+o(x^3))^2 - x^2(1-\frac{x^2}{2}+o(x^2))^2}{x^2(x+o(x))^2}\\ &= \frac{(x^2-\frac{x^4}{3}+o(x^4)) - x^2(1-x^2+o(x^2))}{x^2(x^2+o(x^2))} = \frac{\frac{2}{3}x^4+o(x^4)}{x^4+o(x^4)} = \frac{\frac{2}{3}+o(1)}{1+o(1)} \to \boxed{\frac{2}{3}}\quad (x\to 0).\end{aligned}$$

(ii) 分母 $x(1-\cos x)$ の漸近展開は x^3 の項から始まるから, 分子を $o(x^3)$ を用いて表せばよい.

$$\frac{\log(1+x)-x\sqrt{1-x}}{x(1-\cos x)} = \frac{(x-\frac{1}{2}x^2+\frac{1}{3}x^3+o(x^3))-x(1-\frac{1}{2}x-\frac{1}{8}x^2+o(x^2))}{x(1-1+\frac{1}{2}x^2+o(x^2))}$$
$$= \frac{\frac{11}{24}x^3+o(x^3)}{\frac{1}{2}x^3+o(x^3)} = \frac{\frac{11}{12}+o(1)}{1+o(1)} \to \boxed{\frac{11}{12}} \quad (x\to 0).$$

(2) $x\neq 0$ のとき $x^{\frac{1}{3}}(3-x)^{\frac{2}{3}} = x\Big(\dfrac{3}{x}-1\Big)^{\frac{2}{3}} = x\Big(1-\dfrac{3}{x}\Big)^{\frac{2}{3}}$ と変形できる. ここで, $t=-3/x$ と考え, $t\to 0$ において $(1+t)^{\frac{2}{3}}$ を漸近展開すると, (d) ($\alpha=\frac{2}{3}$) より, $(1+t)^{\frac{2}{3}} = 1+\frac{2}{3}t+\frac{1}{2}\frac{2}{3}(-\frac{1}{3})t^2+o(t^2) = 1+\frac{2}{3}t-\frac{1}{9}t^2+o(t^2)$. よって,

$$x^{\frac{1}{3}}(3-x)^{\frac{2}{3}} = x\Big\{1+\frac{2}{3}\Big(-\frac{3}{x}\Big)-\frac{1}{9}\Big(-\frac{3}{x}\Big)^2+o\Big(\frac{1}{x^2}\Big)\Big\} = x-2-\frac{1}{x}+o\Big(\frac{1}{x}\Big) \quad (x\to\pm\infty).$$

これより, $y=x^{\frac{1}{3}}(3-x)^{\frac{2}{3}}$ の $x\to\pm\infty$ での漸近線は $\boxed{y=x-2}$ であることが分かる.

(3) $\log(1+x)^{\frac{1}{x}} = \dfrac{\log(1+x)}{x} = 1-\dfrac{x}{2}+\dfrac{x^2}{3}+o(x^2)$ であるから, $(1+x)^{\frac{1}{x}} = e^{\log(1+x)^{\frac{1}{x}}} = e^{1-\frac{x}{2}+\frac{x^2}{3}+o(x^2)} = e\cdot e^{-\frac{x}{2}+\frac{x^2}{3}+o(x^2)} =$ $e\Big\{1+\Big(-\dfrac{x}{2}+\dfrac{x^2}{3}\Big)+\dfrac{1}{2}\Big(-\dfrac{x}{2}\Big)^2+o(x^2)\Big\} = e-\dfrac{e}{2}x+\dfrac{11e}{24}x^2+o(x^2)$. 故に, $\Big(1+\dfrac{1}{n}\Big)^n = \boxed{e-\dfrac{e}{2n}+\dfrac{11e}{24n^2}+o\Big(\dfrac{1}{n^2}\Big)}$ $(n\to\infty)$.
同様にして, $\Big(1-\dfrac{1}{n}\Big)^{-n} = e+\dfrac{e}{2n}+\dfrac{11e}{24n^2}+o\Big(\dfrac{1}{n^2}\Big)$ $(n\to\infty)$.

$\boxed{3}$ (1) (前半の不定積分に対する積分定数は省略する.)

(i) 被積分関数を部分分数分解して, $\displaystyle\int\frac{dx}{x(x^2+1)} = \int\Big(\frac{1}{x}-\frac{x}{x^2+1}\Big)dt = \log|x|-\frac{1}{2}\log(x^2+1) = \boxed{\frac{1}{2}\log\frac{x^2}{x^2+1}}$.

(ii) 部分積分法により, $\displaystyle\int x^2\log x\,dx = \frac{x^3}{3}\log x-\int\frac{x^3}{3}\frac{1}{x}\,dx = \frac{x^3\log x}{3}-\int\frac{x^2}{3}dx = \frac{x^3\log x}{3}-\frac{x^3}{9} = \boxed{\frac{x^3}{9}(3\log x-1)}$.

(iii) $x^2=t$ とおけば $x\,dx=\dfrac{dt}{2}$ であるから,

$$\int x^3e^{-x^2}\,dx = \int te^{-t}\frac{dt}{2} = \frac{1}{2}\Big(-te^{-t}+\int e^{-t}\,dt\Big) = -\frac{1}{2}(t+1)e^{-t} = \boxed{-\frac{1}{2}(x^2+1)e^{-x^2}}.$$

(iv) $\cos x=t$ とおけば $-\sin x\,dx=dt$ であるから,

$$\int\frac{dx}{\sin x} = \int\frac{\sin x}{\sin^2 x}\,dx = \int\frac{\sin x}{1-\cos^2 x}\,dx = \int\frac{-dt}{1-t^2} = -\frac{1}{2}\int\Big(\frac{1}{1+t}+\frac{1}{1-t}\Big)dt = \frac{1}{2}\log\Big|\frac{1-t}{1+t}\Big| = \boxed{\frac{1}{2}\log\frac{1-\cos x}{1+\cos x}}.$$

更に, 半角の公式を用いれば, $\displaystyle\int\frac{dx}{\sin x} = \frac{1}{2}\log\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}} = \boxed{\log\Big|\tan\frac{x}{2}\Big|}$.

(v) 部分積分法により, $\displaystyle\int_0^{\frac{\pi}{2}}x^2\cos x\,dx = \Big[x^2\sin x\Big]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}x\sin x\,dx = \frac{\pi^2}{4}+2\Big[x\cos x\Big]_0^{\frac{\pi}{2}}-2\int_0^{\frac{\pi}{2}}\cos x\,dx = \boxed{\frac{\pi^2}{4}-2}$.

(vi) 部分積分法により, $\displaystyle I:=\int_0^{\pi}e^{-x}\sin x\,dx = \Big[-e^{-x}\sin x\Big]_0^{\pi}+\int_0^{\pi}e^{-x}\cos x\,dx = \Big[-e^{-x}\cos x\Big]_0^{\pi}+\int_0^{\pi}e^{-x}(-\sin x)\,dx =$ $1+e^{-\pi}-I$. よって, $I=\boxed{\dfrac{1+e^{-\pi}}{2}}$.

(vii) $$\int_0^{\pi}|\sin x+\cos x|\,dx = \int_0^{\pi}\Big|\sqrt{2}\sin\Big(x+\frac{\pi}{4}\Big)\Big|\,dx = \sqrt{2}\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}|\sin x|\,dx = \sqrt{2}\Big(\int_{\frac{\pi}{4}}^{\pi}\sin x\,dx+\int_{\pi}^{\frac{5\pi}{4}}(-\sin x)\,dx\Big)$$
$$= \sqrt{2}\Big(\Big[-\cos x\Big]_{\frac{\pi}{4}}^{\pi}+\Big[\cos x\Big]_{\pi}^{\frac{5\pi}{4}}\Big) = \sqrt{2}\Big\{\Big(1+\frac{1}{\sqrt{2}}\Big)+\Big(-\frac{1}{\sqrt{2}}+1\Big)\Big\} = \boxed{2\sqrt{2}}.$$

【別法】 $|\sin x|$ は周期 π の関数であるから, $\displaystyle\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}}|\sin x|\,dx = \int_0^{\pi}|\sin x|\,dx = \int_0^{\pi}\sin x\,dx = 2.$

(viii) $\displaystyle I_{m,n} = \int_0^{\pi}(\sin mx)(\cos nx)\,dx$ とおく (m,n は自然数). $m=n$ のとき, $\displaystyle I_{n,n} = \int_0^{\pi}\frac{\sin 2nx}{2}\,dx = \Big[-\frac{\cos 2nx}{4n}\Big]_0^{\pi} = 0.$
また, $m\neq n$ のときは, 三角関数の積和の公式を用いて,

$$I_{m,n} = \frac{1}{2}\int_0^{\pi}\{\sin(m+n)x+\sin(m-n)x\}\,dx = \frac{1}{2}\Big[-\frac{\cos(m+n)x}{m+n}-\frac{\cos(m-n)x}{m-n}\Big]_0^{\pi}$$
$$= \frac{1}{2}\Big\{\frac{1-(-1)^{m+n}}{m+n}+\frac{1-(-1)^{m-n}}{m-n}\Big\} = \boxed{\begin{cases}0 & (m+n\text{ が偶数})\\ \frac{2m}{m^2-n^2} & (m+n\text{ が奇数})\end{cases}}.$$

$m=n$ なら $m+n$ は偶数となるので, $m=n$ の場合も含めて $I_{m,n}$ は, 上の枠内で与えられる.

(2) (i) $c>1$ を定数として, $\displaystyle F(x)=\int_c^x\frac{dt}{\log t}$ $(x>1)$ とおけば, $F'(x)=\dfrac{1}{\log x}$. よって, $f(x)=F(x^3)-F(x^2)$ に合成関数の微分法を適用することにより,

$$f'(x) = F'(x^3)\cdot 3x^2-F'(x^2)\cdot 2x = \frac{1}{\log x^3}\cdot 3x^2-\frac{1}{\log x^2}\cdot 2x = \boxed{\frac{x^2-x}{\log x}}.$$

(ii) $\displaystyle g(x)=x\int_0^x\sin t\,dt-\int_0^x t\sin t\,dt$ であるから, $\displaystyle g'(x)=1\cdot\int_0^x\sin t\,dt+x\cdot\sin x-x\sin x = \Big[-\cos t\Big]_0^x = \boxed{1-\cos x}$.