数学演習第一　演習第 11 回【解答例】

微積：積分の計算 (2)

2021 年 7 月 21 日　実施

1　小テスト問題

問 1　$\displaystyle\int_0^{1/2}\frac{dx}{\sqrt{1-x^2}}=\Big[\mathrm{Sin}^{-1}x\Big]_0^{1/2}=\boxed{\frac{\pi}{6}}$.　(正解は A)

問 2　$\displaystyle\int_0^{\sqrt{3}}\frac{dx}{1+x^2}=\Big[\mathrm{Tan}^{-1}x\Big]_0^{\sqrt{3}}=\boxed{\frac{\pi}{3}}$.　(正解は C)

問 3　$\displaystyle\int_0^1 2^x\,dx=\Big[\frac{2^x}{\log 2}\Big]_0^1=\boxed{\frac{1}{\log 2}}$.　(正解は A)

問 4　$\displaystyle\int_0^1\frac{dx}{\sqrt{x^2+1}}=\Big[\log(x+\sqrt{x^2+1}\,)\Big]_0^1=\boxed{\log(1+\sqrt{2}\,)}$.　(正解は D)

2　レポート課題

問題 1　$\displaystyle I=\int_0^{\sqrt{3}/2}\big(-\sqrt{1-x^2}\,\big)'\,\mathrm{Sin}^{-1}x\,dx=\Big[-\sqrt{1-x^2}\,\mathrm{Sin}^{-1}x\Big]_0^{\sqrt{3}/2}+\int_0^{\sqrt{3}/2}dx=\boxed{-\frac{\pi}{6}+\frac{\sqrt{3}}{2}}$.

《別法》$\theta=\mathrm{Sin}^{-1}x\in(-\pi/2,\pi/2)$ とおけば, $x=\sin\theta$, $\dfrac{dx}{\sqrt{1-x^2}}=d\theta$ より,

$$I=\int_0^{\pi/3}\theta\sin\theta\,d\theta=\Big[-\theta\cos\theta\Big]_0^{\pi/3}+\int_0^{\pi/3}\cos\theta\,d\theta=-\frac{\pi}{6}+\Big[\sin\theta\Big]_0^{\pi/3}=\boxed{\frac{\sqrt{3}}{2}-\frac{\pi}{6}}.$$

問題 2　$\dfrac{2x}{(x+1)(x^2+1)}=\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$ の形に部分分数分解できる．このとき, $2x=a(x^2+1)+(bx+c)(x+1)$ であるから, 両辺の係数を比較して, $a+b=0$, $b+c=2$, $a+c=0$. これより $a=-1$, $b=c=1$ となり, $\dfrac{2x}{(x+1)(x^2+1)}=-\dfrac{1}{x+1}+\dfrac{x+1}{x^2+1}=-\dfrac{1}{x+1}+\dfrac{1}{2}\cdot\dfrac{(x^2+1)'}{x^2+1}+\dfrac{1}{x^2+1}$. よって,

$$J=\int_0^1\frac{2x}{(x+1)(x^2+1)}\,dx=\Big[-\log(x+1)+\frac{1}{2}\log(x^2+1)+\mathrm{Tan}^{-1}x\Big]_0^1=\boxed{-\frac{1}{2}\log 2+\frac{\pi}{4}}.$$

《注》$\displaystyle\int\frac{dx}{x+1}=\log|x+1|$ であるが, $0\le x\le 1$ においては $x+1>0$ であるから, 上では $\log(x+1)$ と書かれている.

問題 3　$t=\sqrt[4]{x}$ とおけば, $x=t^4$, $dx=4t^3\,dt$, $\begin{array}{c|c} x & 0\to 1\\ \hline t & 0\to 1\end{array}$ より,

$$\begin{aligned}K&=\int_0^1\frac{\sqrt[4]{x}}{1+\sqrt{x}}\,dx=\int_0^1\frac{t}{1+t^2}\cdot 4t^3dt=4\int_0^1\frac{t^4}{t^2+1}\,dt\\&=4\int_0^1\Big(t^2-1+\frac{1}{t^2+1}\Big)\,dt=4\Big[\frac{t^3}{3}-t+\mathrm{Tan}^{-1}t\Big]_0^1=4\Big(-\frac{2}{3}+\frac{\pi}{4}\Big)=\boxed{\pi-\frac{8}{3}}.\end{aligned}$$

問題 4　$u=\tan\dfrac{x}{2}$ とおけば, $\sin x=\dfrac{2t}{1+u^2}$, $dx=\dfrac{2\,du}{1+u^2}$, $\begin{array}{c|c} x & 0\to \frac{\pi}{2}\\ \hline u & 0\to 1\end{array}$ より,

$$\begin{aligned}L&=\int_0^{\pi/2}\frac{\sin x}{1+\sin x}\,dx=\int_0^1\frac{\frac{2u}{1+u^2}}{1+\frac{2u}{1+u^2}}\frac{2\,du}{1+u^2}=\int_0^1\frac{4u}{(u+1)^2(u^2+1)}\,du\\&=2\int_0^1\Big\{\frac{1}{u^2+1}-\frac{1}{(u+1)^2}\Big\}\,du=2\Big[\mathrm{Tan}^{-1}u+\frac{1}{u+1}\Big]_0^1=\boxed{\frac{\pi}{2}-1}.\end{aligned}$$

$\dfrac{\sin x}{1+\sin x}=1-\dfrac{1}{1+\sin x}$ と分け, $\displaystyle\int_0^{\pi/2}\frac{dx}{1+\sin x}$ に対して上の置換を適用した方が計算は簡単になる.

《別法》$\dfrac{\sin x}{1+\sin x}=1-\dfrac{1}{1+\sin x}$ と分けて,

$$\int\frac{dx}{1+\sin x}=\int\frac{1-\sin x}{\cos^2 x}\,dx=\tan x-\frac{1}{\cos x}=-\frac{1-\sin x}{\cos x}=-\frac{\cos x}{1+\sin x}\quad\text{(積分定数は省略した)}.$$

よって, $L=\displaystyle\int_0^{\pi/2}\frac{\sin x}{1+\sin x}\,dx=\Big[x+\frac{\cos x}{1+\sin x}\Big]_0^{\pi/2}=\boxed{\frac{\pi}{2}-1}$.

3　演習問題

【注】この解答例では不定積分の積分定数を省略した.

1 前半の (1) から (4) は比較的基本的な不定積分である. ここでの結果は, 通常, 証明せずに用いてよい.

(1) $x=a\tan\theta\ (-\pi/2<\theta<\pi/2)\ \left(\Leftrightarrow\ \theta=\mathrm{Tan}^{-1}\dfrac{x}{a}\right)$ と置換すれば, $dx=\dfrac{a\,d\theta}{\cos^2\theta}$ より, $\displaystyle\int\frac{dx}{x^2+a^2}=\int\frac{1}{a^2(1+\tan^2\theta)}\cdot\frac{a\,d\theta}{\cos^2\theta}=\frac{\theta}{a}=\frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a}$. 《別法》$\displaystyle\int\frac{dx}{x^2+1}=\mathrm{Tan}^{-1}x$ が既知なら, $x=at$ と置換して, $\displaystyle\int\frac{dx}{x^2+a^2}=\frac{1}{a^2}\int\frac{a\,dt}{t^2+1}=\frac{1}{a}\mathrm{Tan}^{-1}t=\frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a}$.

(2) $\dfrac{1}{x^2-a^2}=\dfrac{1}{2a}\left(\dfrac{1}{x-a}-\dfrac{1}{x+a}\right)$ より, $\displaystyle\int\frac{dx}{x^2-a^2}=\frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx=\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$.

(3) $\sqrt{x^2+A}=t-x$ と置換する. 両辺を 2 乗すると x^2 の項が消えて $x=\dfrac{t^2-A}{2t}$ となり, $\dfrac{dx}{dt}=\dfrac{t^2+A}{2t^2}$. よって, $\displaystyle\int\frac{dx}{\sqrt{x^2+A}}=\int\frac{1}{t-\frac{t^2-A}{2t}}\cdot\frac{t^2+A}{2t^2}\,dt=\int\frac{1}{t}\,dt=\log|t|=\log\left|x+\sqrt{x^2+A}\right|$.

(4) $x=a\sin\theta\ (-\pi/2<\theta<\pi/2)\ \left(\Leftrightarrow\ \theta=\mathrm{Sin}^{-1}\dfrac{x}{a}\right)$ と置換すれば, $dx=a\cos\theta\,d\theta$ より, $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\int\frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}\,d\theta=\theta=\mathrm{Sin}^{-1}\frac{x}{a}$. 《別法》$\displaystyle\int\frac{dx}{\sqrt{1-x^2}}=\mathrm{Sin}^{-1}x$ が既知なら, $x=at$ と置換して, $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\frac{1}{a}\int\frac{a\,dt}{\sqrt{1-t^2}}=\mathrm{Sin}^{-1}t=\mathrm{Sin}^{-1}\frac{x}{a}$.

後半の (5) から (8) では $\displaystyle\int f(x)\,dx\left(=\int x'f(x)\,dx\right)=xf(x)-\int xf'(x)\,dx$ を利用する.

(5) まず, $\displaystyle\int\sqrt{x^2+A}\,dx=x\sqrt{x^2+A}-\int x\cdot\frac{x}{\sqrt{x^2+A}}\,dx=x\sqrt{x^2+A}-\int\frac{(x^2+A)-A}{\sqrt{x^2+A}}\,dx=x\sqrt{x^2+A}-\int\sqrt{x^2+A}\,dx+A\int\frac{dx}{\sqrt{x^2+A}}$. よって, **1** (3) を用いて,

$$\int\sqrt{x^2+A}\,dx=\frac{1}{2}\left(x\sqrt{x^2+A}+A\int\frac{dx}{\sqrt{x^2+A}}\right)=\frac{1}{2}\left(x\sqrt{x^2+A}+A\log\left|x+\sqrt{x^2+A}\right|\right).$$

(6) まず, $\displaystyle\int\sqrt{a^2-x^2}\,dx=x\sqrt{a^2-x^2}-\int x\cdot\frac{-x}{\sqrt{a^2-x^2}}\,dx=x\sqrt{a^2-x^2}-\int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\int\frac{dx}{\sqrt{a^2-x^2}}$. よって, **1** (4) を用いて,

$$\int\sqrt{a^2-x^2}\,dx=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\int\frac{dx}{\sqrt{a^2-x^2}}\right)=\frac{1}{2}\left(x\sqrt{a^2-x^2}+a^2\,\mathrm{Sin}^{-1}\frac{x}{a}\right).$$

(7) $\displaystyle\int\mathrm{Sin}^{-1}x\,dx=x\ \mathrm{Sin}^{-1}x-\int\frac{x}{\sqrt{1-x^2}}\,dx=x\ \mathrm{Sin}^{-1}x+\sqrt{1-x^2}$.

(8) $\displaystyle\int\mathrm{Tan}^{-1}x\,dx=x\ \mathrm{Tan}^{-1}x-\int\frac{x}{1+x^2}\,dx=x\ \mathrm{Tan}^{-1}x-\frac{1}{2}\log(1+x^2)$.

2 (1) $\dfrac{x+1}{x^2+2x-63}=\dfrac{x+1}{(x-7)(x+9)}=\dfrac{1}{2}\left(\dfrac{1}{x-7}+\dfrac{1}{x+9}\right)$ なので,

$$\int\frac{x+1}{x^2+2x-63}\,dx=\frac{1}{2}\int\left(\frac{1}{x-7}+\frac{1}{x+9}\right)dx=\frac{1}{2}\big(\log|x-7|+\log|x+9|\big)\,dx=\frac{1}{2}\log|(x-7)(x+9)|.$$

《別法》$\displaystyle\int \frac{x+1}{x^2+2x-63}\,dx = \frac{1}{2}\int \frac{(x^2+2x-63)'}{x^2+2x-63} = \frac{1}{2}\log|x^2+2x-63|$.

(2) $\dfrac{1}{x^4-16} = \dfrac{1}{8}\left(\dfrac{1}{x^2-4} - \dfrac{1}{x^2+4}\right) = \dfrac{1}{8}\left\{\dfrac{1}{4}\left(\dfrac{1}{x-2} - \dfrac{1}{x+2}\right) - \dfrac{1}{x^2+4}\right\}$ と分解できる. $\boxed{1}$(1) を用いて, $\displaystyle\int \frac{dx}{x^2+4} = \int \frac{dx}{x^2+2^2} = \frac{1}{2}\,\mathrm{Tan}^{-1}\,\frac{x}{2}$. よって, $\displaystyle\int \frac{dx}{x^4-16} = \frac{1}{32}\left(\log\left|\frac{x-2}{x+2}\right| - 2\,\mathrm{Tan}^{-1}\,\frac{x}{2}\right)$.

(3) $\dfrac{2x^2+1}{x^2+2} = \dfrac{2(x^2+2)-3}{x^2+2} = 2 - \dfrac{3}{x^2+2}$ なので, $\displaystyle\int \frac{2x^2+1}{x^2+2}\,dx = \int\left\{2 - \frac{3}{x^2+(\sqrt{2}\,)^2}\right\}dx = 2x - \frac{3}{\sqrt{2}}\,\mathrm{Tan}^{-1}\,\frac{x}{\sqrt{2}}$. 最後の等号は $\boxed{1}$(1) による.

(4) $\dfrac{3x^3+x}{x^2+3} = \dfrac{x(3x^2+1)}{x^2+3} = \dfrac{x\{3(x^2+3)-8\}}{x^2+3} = 3x - \dfrac{8x}{x^2+3} = 3x - \dfrac{4(x^2+3)'}{x^2+3}$ なので, $\displaystyle\int \frac{3x^3+x}{x^2+3}\,dx = \frac{3}{2}x^2 - 4\log(x^2+3)$. 《別法》$t = x^2$ とおけば, $\displaystyle\int \frac{3x^3+x}{x^2+3}\,dx = \frac{1}{2}\int \frac{3t+1}{t+3}\,dt = \frac{1}{2}\int\left(3 - \frac{8}{t+3}\right)dt = \frac{1}{2}\big(3t - 8\log|t+3|\big) = \frac{3x^3}{2} - 4\log(x^2+3)$.

(5) 被積分関数の分母は $x^4+4 = (x^2+2)^2 - 4x^2 = (x^2+2x+2)(x^2-2x+2)$ と因数分解でき,

$$\frac{x^2+2}{x^4+4} = \frac{x^2+2}{(x^2+2x+2)(x^2-2x+2)} = \frac{1}{2}\left(\frac{1}{x^2+2x+2} + \frac{1}{x^2-2x+2}\right).$$

ここで, $\boxed{1}$(1) より, $\displaystyle\int \frac{dx}{x^2 \pm 2x + 2} = \int \frac{dx}{(x\pm 1)^2+1} = \mathrm{Tan}^{-1}(x \pm 1)$ (複号同順) であるから, $\displaystyle\int \frac{x^2+2}{x^4+4}\,dx = \frac{1}{2}\big\{\mathrm{Tan}^{-1}(x+1) + \mathrm{Tan}^{-1}(x-1)\big\}$.

(6) $t = x^2$ と置換すれば, $dt = 2x\,dx$ より, $\displaystyle\int \frac{x(x^2+3)}{(x^2-1)(x^2+1)^2} = \frac{1}{2}\int \frac{t+3}{(t-1)(t+1)^2}\,dt$. ここで, $\dfrac{t+3}{(t-1)(t+1)^2} = \dfrac{a}{t-1} + \dfrac{b}{t+1} + \dfrac{c}{(t+1)^2}$ とおき, $t+3 = a(t+1)^2 + b(t-1)(t+1) + c(t-1)$ の両辺の係数を比較して $a+b=0$, $2a+c=1$, $a-b-c=3$. これより $a=1$, $b=c=-1$ であるから, $\displaystyle\int \frac{x(x^2+3)}{(x^2-1)(x^2+1)^2} = \frac{1}{2}\int\left\{\frac{1}{t-1} - \frac{1}{t+1} - \frac{1}{(t+1)^2}\right\}dt = \frac{1}{2}\left(\log\left|\frac{t-1}{t+1}\right| + \frac{1}{t+1}\right) = \frac{1}{2}\log\frac{|x^2-1|}{x^2+1} + \frac{1}{2(x^2+1)}$.

$\boxed{3}$ (1) $t = \sqrt{1+x}$ と置換すると, $dx = 2t\,dt$ なので, $\displaystyle\int \frac{\sqrt{1+x}}{x}\,dx = \int \frac{2t^2}{t^2-1}\,dt = \int \frac{2(t^2-1)+2}{t^2-1}\,dt = 2t + \int\left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt$. よって, $\displaystyle\int \frac{\sqrt{1+x}}{x}\,dx = 2t + \log\left|\frac{t-1}{t+1}\right| = 2\sqrt{1+x} + \log\left|\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\right|$.

(2) $t = \sqrt{\dfrac{2+x}{2-x}}$ $(-2 < x < 2)$ と置換すると, $x = 2 - \dfrac{4}{t^2+1}$ なので, $dx = \dfrac{8t}{(t^2+1)^2}\,dt$ となる. よって,

$$\begin{aligned}\int \sqrt{\frac{2+x}{2-x}}\,dx &= \int t\cdot\frac{8t}{(t^2+1)^2}\,dt = \int t\left(\frac{-4}{t^2+1}\right)'dt = -\frac{4t}{t^2+1} + 4\int\frac{dt}{t^2+1}\\ &= -\frac{4t}{t^2+1} + 4\,\mathrm{Tan}^{-1}\,t = (x-2)\sqrt{\frac{2+x}{2-x}} + 4\,\mathrm{Tan}^{-1}\sqrt{\frac{2+x}{2-x}} = -\sqrt{4-x^2} + 4\,\mathrm{Tan}^{-1}\sqrt{\frac{2+x}{2-x}}.\end{aligned}$$

《別法》$\displaystyle\int \sqrt{\frac{2+x}{2-x}}\,dx = \int \frac{2+x}{\sqrt{4-x^2}}\,dx = \int\left(\frac{2}{\sqrt{2^2-x^2}} + \frac{x}{\sqrt{4-x^2}}\right)dx = 2\,\mathrm{Sin}^{-1}\,\frac{x}{2} - \sqrt{4-x^2}$.

(3) $\sqrt{ax^2+bx+c} = t - \sqrt{a}\,x$ と置換すると, $x = \dfrac{t^2-c}{2\sqrt{a}\,t+b}$, $dx = \dfrac{2(\sqrt{a}\,t^2+bt+\sqrt{a}\,c)}{(2\sqrt{a}\,t+b)^2}\,dt$. よって,

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \int \frac{1}{\frac{t^2-c}{2\sqrt{a}\,t+b}\left(t - \sqrt{a}\cdot\frac{t^2-c}{2\sqrt{a}\,t+b}\right)}\cdot\frac{2(\sqrt{a}\,t^2+bt+\sqrt{a}\,c)}{(2\sqrt{a}\,t+b)^2}\,dt = 2\int\frac{dt}{t^2-c}.$$

$c > 0$ のとき $\displaystyle\int \frac{dt}{t^2-c} = \frac{1}{2\sqrt{c}}\log\left|\frac{t-\sqrt{c}}{t+\sqrt{c}}\right|$, $c = 0$ のとき $\displaystyle\int \frac{dt}{t^2-c} = -\frac{1}{t}$, $c < 0$ のとき $\displaystyle\int \frac{dt}{t^2-c} =$

$\dfrac{1}{\sqrt{|c|}}\,\mathrm{Tan}^{-1}\,\dfrac{t}{\sqrt{|c|}}$ なので,

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{c}}\log\left|\dfrac{\sqrt{ax^2+bx+c}+\sqrt{a}\,x-\sqrt{c}}{\sqrt{ax^2+bx+c}+\sqrt{a}\,x+\sqrt{c}}\right| & (c>0), \\ -\dfrac{2}{\sqrt{ax^2+bx}+\sqrt{a}\,x} & (c=0), \\ \dfrac{2}{\sqrt{|c|}}\,\mathrm{Tan}^{-1}\,\dfrac{\sqrt{ax^2+bx+c}+\sqrt{a}\,x}{\sqrt{|c|}} & (c<0). \end{cases}$$

4 (1) $\sin^2 x = \dfrac{1-\cos 2x}{2}$ なので, $\sin^4 x = \dfrac{1}{4}(1-2\cos 2x+\cos^2 2x) = \dfrac{1}{4}\Big(1-2\cos 2x+\dfrac{1+\cos 4x}{2}\Big) = \dfrac{1}{4}\Big(\dfrac{3}{2}-2\cos 2x+\dfrac{1}{2}\cos 4x\Big)$. よって, $\displaystyle\int \sin^4 x\,dx = \frac{3}{8}x-\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x$.

(2) $u=\tan\dfrac{x}{2}$ と置換すると, $\cos x = \dfrac{1-u^2}{1+u^2}$, $dx = \dfrac{2\,du}{1+u^2}$ より $\displaystyle\int \frac{dx}{4+3\cos x} = \int \frac{1}{4+3\cdot\frac{1-u^2}{1+u^2}}\cdot\frac{2\,du}{1+u^2} = 2\int \frac{du}{7+u^2} = \frac{2}{\sqrt{7}}\,\mathrm{Tan}^{-1}\,\frac{u}{\sqrt{7}}$. よって, $\displaystyle\int \frac{dx}{4+3\cos x} = \frac{2}{\sqrt{7}}\,\mathrm{Tan}^{-1}\Big(\frac{1}{\sqrt{7}}\tan\frac{x}{2}\Big)$.

(3) $u=\tan x$ と置換すると, $\cos^2 x = \dfrac{1}{1+u^2}$, $\sin^2 x = \dfrac{u^2}{1+u^2}$, $dx = \dfrac{du}{1+u^2}$ なので,

$$\int \frac{dx}{\cos^2 x+4\sin^2 x} = \int \frac{1}{\frac{1}{1+u^2}+4\cdot\frac{u^2}{1+u^2}}\cdot\frac{du}{1+u^2} = \int \frac{dt}{1+4u^2} = \frac{1}{4}\int \frac{du}{(\frac{1}{2})^2+u^2}$$
$$= \frac{1}{2}\,\mathrm{Tan}^{-1}\,2u = \frac{1}{2}\,\mathrm{Tan}^{-1}(2\tan x).$$

5 (1) $t=\sqrt{x-1}$ と置換すると, $x=t^2+1$, $dx=2t\,dt$ なので,

$$\int \frac{dx}{x+\sqrt{x-1}} = \int \frac{2t}{t^2+t+1}\,dt = \int \frac{(2t+1)-1}{t^2+t+1}\,dt = \int \frac{(t^2+t+1)'}{t^2+t+1}\,dt - \int \frac{dt}{(t+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$
$$= \log(t^2+t+1) - \frac{1}{\frac{\sqrt{3}}{2}}\,\mathrm{Tan}^{-1}\,\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \log(t^2+t+1) - \frac{2}{\sqrt{3}}\,\mathrm{Tan}^{-1}\,\frac{2t+1}{\sqrt{3}}.$$

よって, $\displaystyle\int_1^2 \frac{dx}{x+\sqrt{x-1}} = \int_0^1 \frac{2t}{t^2+t+1}\,dt = \Big[\log(t^2+t+1)-\frac{2}{\sqrt{3}}\,\mathrm{Tan}^{-1}\,\frac{2t+1}{\sqrt{3}}\Big]_0^1 = \log 3 - \frac{\pi}{3\sqrt{3}}$.

(2) $\displaystyle\int_0^1 \frac{\mathrm{Tan}^{-1}\,x}{1+x^2}\,dx = \frac{1}{2}\int_0^1 \{(\mathrm{Tan}^{-1}\,x)^2\}'\,dx = \frac{1}{2}\Big[\,(\mathrm{Tan}^{-1}\,x)^2\,\Big]_0^1 = \frac{\pi^2}{32}$.

(3) $u=\tan\dfrac{x}{2}$ と置換すると, $\sin x = \dfrac{2u}{1+u^2}$, $dx = \dfrac{2\,du}{1+u^2}$ なので,

$$\int_0^{\pi/2} \frac{dx}{4+5\sin x} = \int_0^1 \frac{1}{4+5\cdot\frac{2u}{1+u^2}}\cdot\frac{2\,du}{1+u^2} = \int_0^1 \frac{dt}{2u^2+5u+2} = \frac{1}{3}\int_0^1 \Big(\frac{2}{2u+1}-\frac{1}{u+2}\Big)\,du$$
$$= \frac{1}{3}\Big[\log\frac{2u+1}{u+2}\Big]_0^1 = \frac{1}{3}\log 2.$$

6 (1) $0<x\le\dfrac{\pi}{2}$ において, $\dfrac{2}{\pi}x\le\sin x\le x$ なので, $\dfrac{1}{x^p}\le\dfrac{1}{(\sin x)^p}\le\Big(\dfrac{\pi}{2}\Big)^p\dfrac{1}{x^p}$. $0<\varepsilon<1$ に対して

$$\int_\varepsilon^{\frac{\pi}{2}} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p}\{(\frac{\pi}{2})^{1-p}-\varepsilon^{1-p}\} & (0<p\ne 1) \\ \log\frac{\pi}{2\varepsilon} & (p=1) \end{cases} \text{ より, } \lim_{\varepsilon\to+0}\int_\varepsilon^{\frac{\pi}{2}} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p}(\frac{\pi}{2})^{1-p} & (0<p<1) \\ \infty & (p\ge 1) \end{cases}$$

となる. よって, 広義積分 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{(\sin x)^p}$ は $0<p<1$ のとき収束し, $p\ge 1$ のとき (∞ に) 発散する.

(2) $(x-a)^2+y^2 \leq b^2 \iff a-\sqrt{b^2-y^2} \leq x \leq a+\sqrt{b^2-y^2}$ より,

$$\pi^{-1}V = \int_{-b}^{b} \{(a+\sqrt{b^2-y^2}\,)^2-(a-\sqrt{b^2-y^2}\,)^2\}\,dy = \int_{-b}^{b} 2a\cdot 2\sqrt{b^2-y^2}\,dy$$

$$= 8a\int_0^b \sqrt{b^2-y^2}\,dy = 2\pi ab^2. \qquad \therefore\ V = 2\pi^2ab^2.$$

(3) L を求めるためには, 第 1 象限の部分の長さを 4 倍すればよい. 第 1 象限の部分は $x=\cos^3 t$, $y=\sin^3 t$ $(0 \leq t \leq \pi/2)$ とパラメータ表示でき, $\dfrac{dx}{dt}=-3\cos^2 t\sin t$, $\dfrac{dy}{dt}=3\sin^2 t\cos t$ となるから,

$$L = 4\int_0^{\frac{\pi}{2}} \sqrt{(-3\cos^2 t\sin t)^2+(3\sin^2 t\cos t)^2}\,dt = 12\int_0^{\frac{\pi}{2}} \cos t\sin t\,dt = 12\left[\frac{1}{2}\sin^2 t\right]_0^{\frac{\pi}{2}} = 6.$$