数学演習第一　演習第 11 回【解答例】

微積：積分の計算 (2) (2024 年 7 月 17 日実施)

1　演習問題

【注】この解答例では不定積分の積分定数を省略した.

$\boxed{1}$ 前半の (1) から (4) は比較的基本的な不定積分である. ここでの結果は, 通常, 証明せずに用いてよい.

(1) $x = a\tan\theta\ (-\pi/2 < \theta < \pi/2)$ $\left(\Leftrightarrow\ \theta = \mathrm{Tan}^{-1}\dfrac{x}{a}\right)$ と置換すれば, $dx = \dfrac{a\,d\theta}{\cos^2\theta}$ より, $\displaystyle\int \frac{dx}{x^2+a^2} = \int \frac{1}{a^2(1+\tan^2\theta)}\cdot\frac{a}{\cos^2\theta}\,d\theta = \frac{\theta}{a} = \boxed{\frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a}}$. 《別法》$\displaystyle\int\frac{dx}{x^2+1} = \mathrm{Tan}^{-1}x$ が既知なら, $x = at$ と置換して, $\displaystyle\int\frac{dx}{x^2+a^2} = \frac{1}{a^2}\int\frac{a\,dt}{t^2+1} = \frac{1}{a}\mathrm{Tan}^{-1}t = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a}$.

(2) $\dfrac{1}{x^2-a^2} = \dfrac{1}{2a}\left(\dfrac{1}{x-a} - \dfrac{1}{x+a}\right)$ より, $\displaystyle\int\frac{dx}{x^2-a^2} = \frac{1}{2a}\int\left(\frac{1}{x-a}-\frac{1}{x+a}\right)dx = \boxed{\frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|}$.

(3) $\sqrt{x^2+A} = t - x$ と置換する. 両辺を 2 乗すると x^2 の項が消えて $x = \dfrac{t^2-A}{2t}$ となり, $\dfrac{dx}{dt} = \dfrac{t^2+A}{2t^2}$. よって, $\displaystyle\int\frac{dx}{\sqrt{x^2+A}} = \int\frac{1}{t-\frac{t^2-A}{2t}}\cdot\frac{t^2+A}{2t^2}\,dt = \int\frac{1}{t}\,dt = \log|t| = \boxed{\log\left|x+\sqrt{x^2+A}\right|}$.

(4) $x = a\sin\theta\ (-\pi/2 < \theta < \pi/2)$ $\left(\Leftrightarrow\ \theta = \mathrm{Sin}^{-1}\dfrac{x}{a}\right)$ と置換すれば, $dx = a\cos\theta\,d\theta$ より, $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}} = \int\frac{a\cos\theta}{a\sqrt{1-\sin^2\theta}}\,d\theta = \theta = \mathrm{Sin}^{-1}\frac{x}{a}$. 《別法》$\displaystyle\int\frac{dx}{\sqrt{1-x^2}} = \mathrm{Sin}^{-1}x$ が既知なら, $x = at$ と置換して, $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int\frac{a\,dt}{\sqrt{1-t^2}} = \mathrm{Sin}^{-1}t = \boxed{\mathrm{Sin}^{-1}\frac{x}{a}}$.

後半の (5) から (8) では $\displaystyle\int f(x)\,dx\left(=\int x'f(x)\,dx\right) = xf(x) - \int xf'(x)\,dx$ を利用する.

(5) まず, $\displaystyle\int\sqrt{x^2+A}\,dx = x\sqrt{x^2+A} - \int x\cdot\frac{x}{\sqrt{x^2+A}}\,dx = x\sqrt{x^2+A} - \int\frac{(x^2+A)-A}{\sqrt{x^2+A}}\,dx = x\sqrt{x^2+A} - \int\sqrt{x^2+A}\,dx + A\int\frac{dx}{\sqrt{x^2+A}}$. よって, $\boxed{1}$(3) を用いて,

$$\int\sqrt{x^2+A}\,dx = \frac{1}{2}\left(x\sqrt{x^2+A} + A\int\frac{dx}{\sqrt{x^2+A}}\right) = \boxed{\frac{1}{2}\left(x\sqrt{x^2+A} + A\log\left|x+\sqrt{x^2+A}\right|\right)}.$$

(6) まず, $\displaystyle\int\sqrt{a^2-x^2}\,dx = x\sqrt{a^2-x^2} - \int x\cdot\frac{-x}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} - \int\frac{(a^2-x^2)-a^2}{\sqrt{a^2-x^2}}\,dx = x\sqrt{a^2-x^2} - \int\sqrt{a^2-x^2}\,dx + a^2\int\frac{dx}{\sqrt{a^2-x^2}}$. よって, $\boxed{1}$(4) を用いて,

$$\int\sqrt{a^2-x^2}\,dx = \frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\int\frac{dx}{\sqrt{a^2-x^2}}\right) = \boxed{\frac{1}{2}\left(x\sqrt{a^2-x^2} + a^2\,\mathrm{Sin}^{-1}\frac{x}{a}\right)}.$$

(7) $\displaystyle\int\mathrm{Sin}^{-1}x\,dx = x\ \mathrm{Sin}^{-1}x - \int\frac{x}{\sqrt{1-x^2}}\,dx = \boxed{x\ \mathrm{Sin}^{-1}x + \sqrt{1-x^2}}$.

(8) $\displaystyle\int\mathrm{Tan}^{-1}x\,dx = x\ \mathrm{Tan}^{-1}x - \int\frac{x}{1+x^2}\,dx = \boxed{x\ \mathrm{Tan}^{-1}x - \frac{1}{2}\log(1+x^2)}$.

$\boxed{2}$ (1) $\dfrac{x-3}{x^2-3x+2} = \dfrac{x-3}{(x-1)(x-2)} = \dfrac{2}{x-1} - \dfrac{1}{x-2}$ なので,

$$\int\frac{x-3}{x^2-3x+2}\,dx = \int\left(\frac{2}{x-1} - \frac{1}{x-2}\right)dx = 2\log|x-1| - \log|x-2| = \boxed{\log\frac{(x-1)^2}{|x-2|}}.$$

(2) $\dfrac{2x}{(x+1)(x^2+1)}=\dfrac{a}{x+1}+\dfrac{bx+c}{x^2+1}$ の形に部分分数分解できる．このとき, $2x=a(x^2+1)+(bx+c)(x+1)$ であるから, 両辺の係数を比較して, $a+b=0$, $b+c=2$, $a+c=0$. これより $a=-1$, $b=c=1$ となり, $\dfrac{2x}{(x+1)(x^2+1)}=-\dfrac{1}{x+1}+\dfrac{x+1}{x^2+1}=-\dfrac{1}{x+1}+\dfrac{1}{2}\cdot\dfrac{(x^2+1)'}{x^2+1}+\dfrac{1}{x^2+1}$. よって,

$$\int\frac{2x}{(x+1)(x^2+1)}\,dx=\boxed{-\log|x+1|+\frac{1}{2}\log(x^2+1)+\mathrm{Tan}^{-1}\,x}.$$

(3) $\dfrac{1}{x^4-16}=\dfrac{1}{8}\left(\dfrac{1}{x^2-4}-\dfrac{1}{x^2+4}\right)=\dfrac{1}{8}\left\{\dfrac{1}{4}\left(\dfrac{1}{x-2}-\dfrac{1}{x+2}\right)-\dfrac{1}{x^2+4}\right\}$ と分解できる．$\boxed{1}$ (1) を用いて, $\displaystyle\int\frac{dx}{x^2+4}=\int\frac{dx}{x^2+2^2}=\frac{1}{2}\,\mathrm{Tan}^{-1}\,\frac{x}{2}$. よって, $\displaystyle\int\frac{dx}{x^4-16}=\boxed{\frac{1}{32}\left(\log\left|\frac{x-2}{x+2}\right|-2\,\mathrm{Tan}^{-1}\,\frac{x}{2}\right)}$.

(4) $\dfrac{3x^3+x}{x^2+3}=\dfrac{x(3x^2+1)}{x^2+3}=\dfrac{x\{3(x^2+3)-8\}}{x^2+3}=3x-\dfrac{8x}{x^2+3}=3x-\dfrac{4(x^2+3)'}{x^2+3}$ なので, $\displaystyle\int\frac{3x^3+x}{x^2+3}\,dx=$ $\boxed{\dfrac{3}{2}x^2-4\log(x^2+3)}$. 《別法》$t=x^2$ とおけば, $\displaystyle\int\frac{3x^3+x}{x^2+3}\,dx=\frac{1}{2}\int\frac{3t+1}{t+3}\,dt=\frac{1}{2}\int\left(3-\frac{8}{t+3}\right)dt=$ $\dfrac{1}{2}\bigl(3t-8\log|t+3|\bigr)=\dfrac{3}{2}x^3-4\log(x^2+3)$.

(5) 被積分関数の分母は $x^4+4=(x^2+2)^2-4x^2=(x^2+2x+2)(x^2-2x+2)$ と因数分解でき,

$$\frac{x^2+2}{x^4+4}=\frac{x^2+2}{(x^2+2x+2)(x^2-2x+2)}=\frac{1}{2}\left(\frac{1}{x^2+2x+2}+\frac{1}{x^2-2x+2}\right).$$

ここで, $\boxed{1}$ (1) より, $\displaystyle\int\frac{dx}{x^2\pm 2x+2}=\int\frac{dx}{(x\pm1)^2+1}=\mathrm{Tan}^{-1}(x\pm1)$ (複号同順) であるから, $\displaystyle\int\frac{x^2+2}{x^4+4}\,dx=\boxed{\frac{1}{2}\bigl\{\mathrm{Tan}^{-1}(x+1)+\mathrm{Tan}^{-1}(x-1)\bigr\}}$.

(6) $t=x^2$ と置換すれば, $dt=2x\,dx$ より, $\displaystyle\int\frac{x(x^2+3)}{(x^2-1)(x^2+1)^2}=\frac{1}{2}\int\frac{t+3}{(t-1)(t+1)^2}\,dt$. ここで, $\dfrac{t+3}{(t-1)(t+1)^2}=\dfrac{a}{t-1}+\dfrac{b}{t+1}+\dfrac{c}{(t+1)^2}$ とおき, $t+3=a(t+1)^2+b(t-1)(t+1)+c(t-1)$ の両辺の係数を比較して $a+b=0$, $2a+c=1$, $a-b-c=3$. これより $a=1$, $b=c=-1$ であるから, $\displaystyle\int\frac{x(x^2+3)}{(x^2-1)(x^2+1)^2}=\frac{1}{2}\int\left\{\frac{1}{t-1}-\frac{1}{t+1}-\frac{1}{(t+1)^2}\right\}dt=\frac{1}{2}\left(\log\left|\frac{t-1}{t+1}\right|+\frac{1}{t+1}\right)=$ $\boxed{\dfrac{1}{2}\log\dfrac{|x^2-1|}{x^2+1}+\dfrac{1}{2(x^2+1)}}$.

$\boxed{3}$ (1) $t=\sqrt{x-1}$ と置換すると, $x=1+t^2$, $dx=2t\,dt$ なので, $\displaystyle\int\frac{\sqrt{x-1}}{x}\,dx=\int\frac{t}{1+t^2}\cdot 2t\,dt$

$$=\int\frac{2t^2}{1+t^2}dt=2\int\left(1-\frac{1}{1+t^2}\right)dt=2t-2\,\mathrm{Tan}^{-1}\,t=\boxed{2\sqrt{x-1}-2\,\mathrm{Tan}^{-1}\sqrt{x-1}}.$$

(2) $t=\sqrt{\dfrac{2+x}{2-x}}$ $(-2<x<2)$ と置換すると, $x=2-\dfrac{4}{t^2+1}$ なので, $dx=\dfrac{8t}{(t^2+1)^2}\,dt$ となる．よって,

$$\int\sqrt{\frac{2+x}{2-x}}\,dx=\int t\cdot\frac{8t}{(t^2+1)^2}\,dt=\int t\left(\frac{-4}{t^2+1}\right)'dt=-\frac{4t}{t^2+1}+4\int\frac{dt}{t^2+1}$$

$$=-\frac{4t}{t^2+1}+4\,\mathrm{Tan}^{-1}\,t=(x-2)\sqrt{\frac{2+x}{2-x}}+4\,\mathrm{Tan}^{-1}\sqrt{\frac{2+x}{2-x}}=\boxed{-\sqrt{4-x^2}+4\,\mathrm{Tan}^{-1}\sqrt{\frac{2+x}{2-x}}}.$$

《別法》$\displaystyle\int\sqrt{\frac{2+x}{2-x}}\,dx=\int\frac{2+x}{\sqrt{4-x^2}}\,dx=\int\left(\frac{2}{\sqrt{2^2-x^2}}+\frac{x}{\sqrt{4-x^2}}\right)dx=2\,\mathrm{Sin}^{-1}\,\frac{x}{2}-\sqrt{4-x^2}$.

ここで, $\mathrm{Tan}^{-1}\sqrt{\dfrac{2+x}{2-x}}=\dfrac{1}{2}\left(\mathrm{Sin}^{-1}\,\dfrac{x}{2}+\dfrac{\pi}{2}\right)$ だから, これは上の結果と定数の差を除いて一致している．

(3) $\sqrt{ax^2+bx+c}=t-\sqrt{a}\,x$ と置換すると, $x=\dfrac{t^2-c}{2\sqrt{a}\,t+b}$, $dx=\dfrac{2(\sqrt{a}\,t^2+bt+\sqrt{a}\,c)}{(2\sqrt{a}\,t+b)^2}\,dt$. よって,

$$\int\frac{dx}{x\sqrt{ax^2+bx+c}}=\int\frac{1}{\frac{t^2-c}{2\sqrt{a}\,t+b}\left(t-\sqrt{a}\cdot\frac{t^2-c}{2\sqrt{a}\,t+b}\right)}\cdot\frac{2(\sqrt{a}\,t^2+bt+\sqrt{a}\,c)}{(2\sqrt{a}\,t+b)^2}\,dt=2\int\frac{dt}{t^2-c}.$$

$c>0$ のとき $\displaystyle\int\frac{dt}{t^2-c}=\frac{1}{2\sqrt{c}}\log\left|\frac{t-\sqrt{c}}{t+\sqrt{c}}\right|$, $c=0$ のとき $\displaystyle\int\frac{dt}{t^2-c}=-\frac{1}{t}$, $c<0$ のとき $\displaystyle\int\frac{dt}{t^2-c}=\frac{1}{\sqrt{|c|}}\,\mathrm{Tan}^{-1}\,\frac{t}{\sqrt{|c|}}$ なので,

$$\int\frac{dx}{x\sqrt{ax^2+bx+c}}=\boxed{\begin{cases}\dfrac{1}{\sqrt{c}}\log\left|\dfrac{\sqrt{ax^2+bx+c}+\sqrt{a}\,x-\sqrt{c}}{\sqrt{ax^2+bx+c}+\sqrt{a}\,x+\sqrt{c}}\right| & (c>0),\\ -\dfrac{2}{\sqrt{ax^2+bx}+\sqrt{a}\,x} & (c=0),\\ \dfrac{2}{\sqrt{|c|}}\,\mathrm{Tan}^{-1}\,\dfrac{\sqrt{ax^2+bx+c}+\sqrt{a}\,x}{\sqrt{|c|}} & (c<0).\end{cases}}$$

4 (1) $u=\tan\dfrac{x}{2}$ と置換すると. $\cos x=\dfrac{1-u^2}{1+u^2}$, $\sin x=\dfrac{2u}{1+u^2}$, $dx=\dfrac{2\,du}{1+u^2}$ より,

$$\int\frac{1+\sin x}{\sin x(1+\cos x)}dx=\int\frac{1+\frac{2u}{1+u^2}}{\frac{2u}{1+u^2}(1+\frac{1-u^2}{1+u^2})}\cdot\frac{2}{1+u^2}du=\int\frac{u^2+2u+1}{2u}du=\frac{1}{2}\int\left(u+2+\frac{1}{u}\right)du$$
$$=\frac{1}{4}u^2+u+\frac{1}{2}\log|u|=\boxed{\frac{1}{4}\tan^2\frac{x}{2}+\tan\frac{x}{2}+\frac{1}{2}\log\left|\tan\frac{x}{2}\right|}.$$

(2) $\sin^2 x=\dfrac{1-\cos 2x}{2}$, $\cos^2 x=\dfrac{1+\cos 2x}{2}$ より, $\sin^4 x+\cos^4 x=\dfrac{1}{2}+\dfrac{1}{2}\cos^2 2x=\dfrac{3}{4}+\dfrac{1}{4}\cos 4x$.
よって, $\displaystyle\int(\sin^4 x+\cos^4 x)\,dx=\boxed{\frac{3}{4}x+\frac{1}{16}\sin 4x}$.

(3) $u=\tan x$ と置換すると, $\cos^2 x=\dfrac{1}{1+u^2}$, $\sin^2 x=\dfrac{u^2}{1+u^2}$, $dx=\dfrac{du}{1+u^2}$ なので,

$$\int\frac{dx}{\cos^2 x+4\sin^2 x}=\int\frac{1}{\frac{1}{1+u^2}+4\cdot\frac{u^2}{1+u^2}}\cdot\frac{du}{1+u^2}=\int\frac{dt}{1+4u^2}=\frac{1}{4}\int\frac{du}{(\frac{1}{2})^2+u^2}$$
$$=\frac{1}{2}\,\mathrm{Tan}^{-1}\,2u=\boxed{\frac{1}{2}\,\mathrm{Tan}^{-1}(2\tan x)}.$$

5 (1) $t=\sqrt{1-x}$ と置換すると, $x=1-t^2$, $dx=-2t\,dt$ なので,

$$\int_0^1\frac{dx}{1+x+\sqrt{1-x}}=\int_1^0\frac{-2t}{2-t^2+t}dt=-\int_0^1\frac{2t}{(t-2)(t+1)}dt=-\frac{2}{3}\int_0^1\left(\frac{2}{t-2}+\frac{1}{t+1}\right)dt$$
$$=-\frac{2}{3}\Big[2\log|t-2|+\log|t+1|\Big]_0^1=\boxed{\frac{2}{3}\log 2}.$$

(2) $\displaystyle\int_0^{\frac{1}{2}}\frac{\mathrm{Sin}^{-1}\,x}{\sqrt{1-x^2}}\,dx=\int_0^{\frac{1}{2}}\mathrm{Sin}^{-1}\,x\,(\mathrm{Sin}^{-1}\,x)'\,dx=\left[\frac{1}{2}\,(\mathrm{Sin}^{-1}\,x)^2\right]_0^{\frac{1}{2}}=\frac{1}{2}\left(\frac{\pi}{6}\right)^2=\boxed{\frac{\pi^2}{72}}$.

(3) $u=\tan\dfrac{x}{2}$ と置換すると, $\sin x=\dfrac{2u}{1+u^2}$, $dx=\dfrac{2\,du}{1+u^2}$ なので,

$$\int_0^{\pi/2}\frac{dx}{4+5\sin x}=\int_0^1\frac{1}{4+5\cdot\frac{2u}{1+u^2}}\cdot\frac{2\,du}{1+u^2}=\int_0^1\frac{du}{2u^2+5u+2}=\int_0^1\frac{du}{(2u+1)(u+2)}$$
$$=\frac{1}{3}\int_0^1\left(\frac{2}{2u+1}-\frac{1}{u+2}\right)du=\frac{1}{3}\left[\log\frac{2u+1}{u+2}\right]_0^1=\boxed{\frac{1}{3}\log 2}.$$

6 (1) $0<x\le\dfrac{\pi}{2}$ において, $\dfrac{2}{\pi}x\le\sin x\le x$ なので, $\dfrac{1}{x^p}\le\dfrac{1}{(\sin x)^p}\le\left(\dfrac{\pi}{2}\right)^p\dfrac{1}{x^p}$. $0<\varepsilon<1$ に対して

$$\int_\varepsilon^{\pi/2}\frac{dx}{x^p}=\begin{cases}\frac{1}{1-p}\{(\frac{\pi}{2})^{1-p}-\varepsilon^{1-p}\} & (0<p\ne 1)\\ \log\frac{\pi}{2\varepsilon} & (p=1)\end{cases}\quad\text{より,}\quad\lim_{\varepsilon\to+0}\int_\varepsilon^{\pi/2}\frac{dx}{x^p}=\begin{cases}\frac{1}{1-p}(\frac{\pi}{2})^{1-p} & (0<p<1)\\ \infty & (p\ge 1)\end{cases}$$

となる．よって，広義積分 $\displaystyle\int_0^{\pi/2} \frac{dx}{(\sin x)^p}$ は $0 < p < 1$ のとき収束し，$p \geq 1$ のとき (∞ に) 発散する．

(2) $(x-a)^2 + y^2 \leq b^2 \ \Leftrightarrow\ a - \sqrt{b^2 - y^2} \leq x \leq a + \sqrt{b^2 - y^2}$ より，

$$\pi^{-1}V = \int_{-b}^{b} \{(a + \sqrt{b^2-y^2}\,)^2 - (a - \sqrt{b^2-y^2}\,)^2\}\,dy = \int_{-b}^{b} 2a \cdot 2\sqrt{b^2-y^2}\,dy$$
$$= 8a\int_0^b \sqrt{b^2-y^2}\,dy = 2\pi ab^2. \qquad \therefore\ V = \boxed{2\pi^2 ab^2}.$$

(3) L を求めるためには，第 1 象限の部分の長さを 4 倍すればよい．第 1 象限の部分は $x = \cos^3 t$, $y = \sin^3 t$ $(0 \leq t \leq \pi/2)$ とパラメータ表示でき，$\dfrac{dx}{dt} = -3\cos^2 t \sin t$, $\dfrac{dy}{dt} = 3\sin^2 t\cos t$ となるから，

$$L = 4\int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2}\,dt = 12\int_0^{\pi/2} \cos t \sin t\,dt = 12\left[\frac{1}{2}\sin^2 t\right]_0^{\pi/2} = \boxed{6}.$$

2 レポート問題

$\boxed{\textbf{1}}$ (1) $t = \sqrt{x}$ と置換すると，$x = t^2, dx = 2t\,dt$ なので，

$$\int \log(1+\sqrt{x})\,dx = \int \log(1+t)\cdot 2t\,dt = t^2\log(1+t) - \int \frac{t^2}{1+t}\,dt$$
$$= t^2\log(1+t) - \int\left(t - 1 + \frac{1}{1+t}\right)dt = t^2\log(1+t) - \frac{t^2}{2} + t - \log(1+t)$$
$$= \boxed{(x-1)\log(1+\sqrt{x}) - \frac{x}{2} + \sqrt{x}}.$$

(2) $\displaystyle\int \frac{x+1}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{(1-x^2)'}{\sqrt{1-x^2}}\,dx + \int \frac{dx}{\sqrt{1-x^2}} = \boxed{-\sqrt{1-x^2} + \mathrm{Sin}^{-1}\,x}.$

(3) $u = \tan\dfrac{x}{2}$ と置換すると，$\cos x = \dfrac{1-u^2}{1+u^2}$，$dx = \dfrac{2}{1+u^2}du$ なので，

$$\int \frac{dx}{2+\cos x}\,dx = \int \frac{1}{2 + \frac{1-u^2}{1+u^2}}\,\frac{2}{1+u^2}\,du = \int \frac{2}{3+u^2}\,du = \frac{2}{\sqrt{3}}\,\mathrm{Tan}^{-1}\,\frac{u}{\sqrt{3}} = \boxed{\frac{2}{\sqrt{3}}\,\mathrm{Tan}^{-1}\Big(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\Big)}.$$

$\boxed{\textbf{2}}$ (1) $t = \sqrt{1-x}$ と置換すると，$x = 1 - t^2$，$dx = -2t\,dt$ なので，

$$\int_0^1 x^2\sqrt{1-x}\,dx = \int_1^0 (1-t^2)^2 t\cdot(-2t)\,dt = 2\int_0^1 (t^6 - 2t^4 + t^2)\,dt = 2\Big[\frac{t^7}{7} - \frac{2t^5}{5} + \frac{t^3}{3}\Big]_0^1 = \boxed{\frac{16}{105}}.$$

(2) $\displaystyle\int_0^1 \frac{dx}{3x^4 + 10x^2 + 3} = \int_0^1 \frac{dx}{(3x^2+1)(x^2+3)} = \frac{1}{8}\int_0^1 \Big(\frac{3}{3x^2+1} - \frac{1}{x^2+3}\Big)\,dx$

$$= \frac{1}{8}\Big[\sqrt{3}\,\mathrm{Tan}^{-1}(\sqrt{3}) - \frac{1}{\sqrt{3}}\,\mathrm{Tan}^{-1}\,\frac{1}{\sqrt{3}}\Big]_0^1 = \frac{1}{8}\Big(\sqrt{3}\cdot\frac{\pi}{3} - \frac{1}{\sqrt{3}}\cdot\frac{\pi}{6}\Big) = \boxed{\frac{5\pi}{48\sqrt{3}}\Big(= \frac{5\sqrt{3}\pi}{144}\Big)}.$$

(3) $\displaystyle\int_{\sqrt{3}/2}^1 \mathrm{Cos}^{-1}x\,dx = \big[x\,\mathrm{Cos}^{-1}x\big]_{\sqrt{3}/2}^1 + \int_{\sqrt{3}/2}^1 \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6} - \big[\sqrt{1-x^2}\big]_{\sqrt{3}/2}^1 = \boxed{\frac{1}{2} - \frac{\sqrt{3}\pi}{12}}.$