

数学演習第一 演習第 11 回【解答例】

微積：積分の計算 (2) (2026 年 7 月 22 日実施)

演習問題

【注】この解答例では不定積分の積分定数を省略した。

1 前半の (1) から (4) は比較的基本的な不定積分である。ここでの結果は、通常、証明せずに用いてよい。

$$(1) \quad x = a \tan \theta \quad (-\pi/2 < \theta < \pi/2) \quad \left(\Leftrightarrow \theta = \tan^{-1} \frac{x}{a} \right) \text{ と置換すれば, } dx = \frac{a d\theta}{\cos^2 \theta} \text{ より, } \int \frac{dx}{x^2 + a^2} = \int \frac{1}{a^2(1 + \tan^2 \theta)} \cdot \frac{a}{\cos^2 \theta} d\theta = \frac{\theta}{a} = \boxed{\frac{1}{a} \tan^{-1} \frac{x}{a}}. \text{ 《別法》 } \int \frac{dx}{x^2 + 1} = \tan^{-1} x \text{ が既知なら, } x = at \text{ と置換して, } \int \frac{dx}{x^2 + a^2} = \frac{1}{a^2} \int \frac{a dt}{t^2 + 1} = \frac{1}{a} \tan^{-1} t = \frac{1}{a} \tan^{-1} \frac{x}{a}.$$

$$(2) \quad \frac{1}{x^2 - a^2} = \frac{1}{2a} \left(\frac{1}{x-a} - \frac{1}{x+a} \right) \text{ より, } \int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \int \left(\frac{1}{x-a} - \frac{1}{x+a} \right) dx = \boxed{\frac{1}{2a} \log \left| \frac{x-a}{x+a} \right|}.$$

$$(3) \quad \sqrt{x^2 + A} = t - x \text{ と置換する. 両辺を 2 乗すると } x^2 \text{ の項が消えて } x = \frac{t^2 - A}{2t} \text{ となり, } \frac{dx}{dt} = \frac{t^2 + A}{2t^2}. \text{ よって, } \int \frac{dx}{\sqrt{x^2 + A}} = \int \frac{1}{t - \frac{t^2 - A}{2t}} \cdot \frac{t^2 + A}{2t^2} dt = \int \frac{1}{t} dt = \log|t| = \boxed{\log|x + \sqrt{x^2 + A}|}.$$

$$(4) \quad x = a \sin \theta \quad (-\pi/2 < \theta < \pi/2) \quad \left(\Leftrightarrow \theta = \sin^{-1} \frac{x}{a} \right) \text{ と置換すれば, } dx = a \cos \theta d\theta \text{ より, } \int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{a \cos \theta}{a \sqrt{1 - \sin^2 \theta}} d\theta = \theta = \sin^{-1} \frac{x}{a}. \text{ 《別法》 } \int \frac{dx}{\sqrt{1 - x^2}} = \sin^{-1} x \text{ が既知なら, } x = at \text{ と置換して, } \int \frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{a} \int \frac{a dt}{\sqrt{1 - t^2}} = \sin^{-1} t = \boxed{\sin^{-1} \frac{x}{a}}.$$

後半の (5) から (8) では $\int f(x) dx \left(= \int x' f(x) dx \right) = x f(x) - \int x f'(x) dx$ を利用する。

$$(5) \quad \text{まず, } \int \sqrt{x^2 + A} dx = x \sqrt{x^2 + A} - \int x \cdot \frac{x}{\sqrt{x^2 + A}} dx = x \sqrt{x^2 + A} - \int \frac{(x^2 + A) - A}{\sqrt{x^2 + A}} dx = x \sqrt{x^2 + A} - \int \sqrt{x^2 + A} dx + A \int \frac{dx}{\sqrt{x^2 + A}}. \text{ よって, } \boxed{1} (3) \text{ を用いて,}$$

$$\int \sqrt{x^2 + A} dx = \frac{1}{2} \left(x \sqrt{x^2 + A} + A \int \frac{dx}{\sqrt{x^2 + A}} \right) = \boxed{\frac{1}{2} \left(x \sqrt{x^2 + A} + A \log|x + \sqrt{x^2 + A}| \right)}.$$

$$(6) \quad \text{まず, } \int \sqrt{a^2 - x^2} dx = x \sqrt{a^2 - x^2} - \int x \cdot \frac{-x}{\sqrt{a^2 - x^2}} dx = x \sqrt{a^2 - x^2} - \int \frac{(a^2 - x^2) - a^2}{\sqrt{a^2 - x^2}} dx = x \sqrt{a^2 - x^2} - \int \sqrt{a^2 - x^2} dx + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}}. \text{ よって, } \boxed{1} (4) \text{ を用いて,}$$

$$\int \sqrt{a^2 - x^2} dx = \frac{1}{2} \left(x \sqrt{a^2 - x^2} + a^2 \int \frac{dx}{\sqrt{a^2 - x^2}} \right) = \boxed{\frac{1}{2} \left(x \sqrt{a^2 - x^2} + a^2 \sin^{-1} \frac{x}{a} \right)}.$$

$$(7) \quad \int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx = \boxed{x \sin^{-1} x + \sqrt{1 - x^2}}.$$

$$(8) \quad \int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x}{1 + x^2} dx = \boxed{x \tan^{-1} x - \frac{1}{2} \log(1 + x^2)}.$$

2 (1) $\frac{x-3}{x^2-3x+2} = \frac{x-3}{(x-1)(x-2)} = \frac{2}{x-1} - \frac{1}{x-2}$ なので,

$$\int \frac{x-3}{x^2-3x+2} dx = \int \left(\frac{2}{x-1} - \frac{1}{x-2} \right) dx = 2 \log|x-1| - \log|x-2| = \boxed{\log \frac{(x-1)^2}{|x-2|}}.$$

- (2) $\frac{2x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$ の形に部分分数分解できる. このとき, $2x = a(x^2+1) + (bx+c)(x+1)$ であるから, 両辺の係数を比較して, $a+b=0$, $b+c=2$, $a+c=0$. これより $a=-1$, $b=c=1$ となり,

$$\frac{2x}{(x+1)(x^2+1)} = -\frac{1}{x+1} + \frac{x+1}{x^2+1} = -\frac{1}{x+1} + \frac{1}{2} \cdot \frac{(x^2+1)'}{x^2+1} + \frac{1}{x^2+1}.$$
 よって,

$$\int \frac{2x}{(x+1)(x^2+1)} dx = \boxed{-\log|x+1| + \frac{1}{2} \log(x^2+1) + \text{Tan}^{-1} x}.$$

- (3) $\frac{1}{x^4-16} = \frac{1}{8} \left(\frac{1}{x^2-4} - \frac{1}{x^2+4} \right) = \frac{1}{8} \left\{ \frac{1}{4} \left(\frac{1}{x-2} - \frac{1}{x+2} \right) - \frac{1}{x^2+4} \right\}$ と分解できる. $\boxed{1}$ (1) を用いて,

$$\int \frac{dx}{x^2+4} = \int \frac{dx}{x^2+2^2} = \frac{1}{2} \text{Tan}^{-1} \frac{x}{2}.$$
 よって,
$$\int \frac{dx}{x^4-16} = \boxed{\frac{1}{32} \left(\log \left| \frac{x-2}{x+2} \right| - 2 \text{Tan}^{-1} \frac{x}{2} \right)}.$$

- (4) $\frac{3x^3+x}{x^2+3} = \frac{x(3x^2+1)}{x^2+3} = \frac{x\{3(x^2+3)-8\}}{x^2+3} = 3x - \frac{8x}{x^2+3} = 3x - \frac{4(x^2+3)'}{x^2+3}$ なので, $\int \frac{3x^3+x}{x^2+3} dx =$

$$\boxed{\frac{3}{2}x^2 - 4 \log(x^2+3)}.$$
 《別法》 $t = x^2$ とおけば, $\int \frac{3x^3+x}{x^2+3} dx = \frac{1}{2} \int \frac{3t+1}{t+3} dt = \frac{1}{2} \int \left(3 - \frac{8}{t+3} \right) dt =$

$$\frac{1}{2} (3t - 8 \log|t+3|) = \frac{3}{2}x^2 - 4 \log(x^2+3).$$

- (5) 被積分関数の分母は $x^4+4 = (x^2+2)^2 - 4x^2 = (x^2+2x+2)(x^2-2x+2)$ と因数分解でき,

$$\frac{x^2+2}{x^4+4} = \frac{x^2+2}{(x^2+2x+2)(x^2-2x+2)} = \frac{1}{2} \left(\frac{1}{x^2+2x+2} + \frac{1}{x^2-2x+2} \right).$$

ここで, $\boxed{1}$ (1) より, $\int \frac{dx}{x^2 \pm 2x + 2} = \int \frac{dx}{(x \pm 1)^2 + 1} = \text{Tan}^{-1}(x \pm 1)$ (複号同順) であるから,

$$\int \frac{x^2+2}{x^4+4} dx = \boxed{\frac{1}{2} \{ \text{Tan}^{-1}(x+1) + \text{Tan}^{-1}(x-1) \}}.$$

- (6) $t = x^2$ と置換すれば, $dt = 2x dx$ より, $\int \frac{x(x^2+3)}{(x^2-1)(x^2+1)^2} = \frac{1}{2} \int \frac{t+3}{(t-1)(t+1)^2} dt.$ ここで,

$$\frac{t+3}{(t-1)(t+1)^2} = \frac{a}{t-1} + \frac{b}{t+1} + \frac{c}{(t+1)^2}$$
 とおき, $t+3 = a(t+1)^2 + b(t-1)(t+1) + c(t-1)$ の両辺の係数を比較して $a+b=0$, $2a+c=1$, $a-b-c=3$. これより $a=1$, $b=c=-1$ であるから,

$$\int \frac{x(x^2+3)}{(x^2-1)(x^2+1)^2} = \frac{1}{2} \int \left\{ \frac{1}{t-1} - \frac{1}{t+1} - \frac{1}{(t+1)^2} \right\} dt = \frac{1}{2} \left(\log \left| \frac{t-1}{t+1} \right| + \frac{1}{t+1} \right) =$$

$$\boxed{\frac{1}{2} \log \frac{|x^2-1|}{x^2+1} + \frac{1}{2(x^2+1)}}.$$

$\boxed{3}$ (1) $t = \sqrt{x-1}$ と置換すると, $x = 1+t^2$, $dx = 2t dt$ なので, $\int \frac{\sqrt{x-1}}{x} dx = \int \frac{t}{1+t^2} \cdot 2t dt$

$$= \int \frac{2t^2}{1+t^2} dt = 2 \int \left(1 - \frac{1}{1+t^2} \right) dt = 2t - 2 \text{Tan}^{-1} t = \boxed{2\sqrt{x-1} - 2 \text{Tan}^{-1} \sqrt{x-1}}.$$

- (2) $t = \sqrt{\frac{2+x}{2-x}}$ ($-2 < x < 2$) と置換すると, $x = 2 - \frac{4}{t^2+1}$ なので, $dx = \frac{8t}{(t^2+1)^2} dt$ となる. よって,

$$\int \sqrt{\frac{2+x}{2-x}} dx = \int t \cdot \frac{8t}{(t^2+1)^2} dt = \int t \left(\frac{-4}{t^2+1} \right)' dt = -\frac{4t}{t^2+1} + 4 \int \frac{dt}{t^2+1}$$

$$= -\frac{4t}{t^2+1} + 4 \text{Tan}^{-1} t = (x-2) \sqrt{\frac{2+x}{2-x}} + 4 \text{Tan}^{-1} \sqrt{\frac{2+x}{2-x}} = \boxed{-\sqrt{4-x^2} + 4 \text{Tan}^{-1} \sqrt{\frac{2+x}{2-x}}}.$$

《別法》 $\int \sqrt{\frac{2+x}{2-x}} dx = \int \frac{2+x}{\sqrt{4-x^2}} dx = \int \left(\frac{2}{\sqrt{4-x^2}} + \frac{x}{\sqrt{4-x^2}} \right) dx = 2 \text{Sin}^{-1} \frac{x}{2} - \sqrt{4-x^2}.$

ここで, $\text{Tan}^{-1} \sqrt{\frac{2+x}{2-x}} = \frac{1}{2} \left(\text{Sin}^{-1} \frac{x}{2} + \frac{\pi}{2} \right)$ だから, これは上の結果と定数の差を除いて一致している.

(3) $\sqrt{ax^2+bx+c}=t-\sqrt{a}x$ と置換すると, $x=\frac{t^2-c}{2\sqrt{a}t+b}$, $dx=\frac{2(\sqrt{a}t^2+bt+\sqrt{a}c)}{(2\sqrt{a}t+b)^2}dt$. よって,

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \int \frac{1}{\frac{t^2-c}{2\sqrt{a}t+b}(t-\sqrt{a}\cdot\frac{t^2-c}{2\sqrt{a}t+b})} \cdot \frac{2(\sqrt{a}t^2+bt+\sqrt{a}c)}{(2\sqrt{a}t+b)^2} dt = 2 \int \frac{dt}{t^2-c}.$$

$c > 0$ のとき $\int \frac{dt}{t^2-c} = \frac{1}{2\sqrt{c}} \log \left| \frac{t-\sqrt{c}}{t+\sqrt{c}} \right|$, $c = 0$ のとき $\int \frac{dt}{t^2-c} = -\frac{1}{t}$, $c < 0$ のとき $\int \frac{dt}{t^2-c} = \frac{1}{\sqrt{|c|}} \text{Tan}^{-1} \frac{t}{\sqrt{|c|}}$ なので,

$$\int \frac{dx}{x\sqrt{ax^2+bx+c}} = \begin{cases} \frac{1}{\sqrt{c}} \log \left| \frac{\sqrt{ax^2+bx+c} + \sqrt{a}x - \sqrt{c}}{\sqrt{ax^2+bx+c} + \sqrt{a}x + \sqrt{c}} \right| & (c > 0), \\ -\frac{1}{\sqrt{ax^2+bx+c} + \sqrt{a}x} & (c = 0), \\ \frac{2}{\sqrt{|c|}} \text{Tan}^{-1} \frac{\sqrt{ax^2+bx+c} + \sqrt{a}x}{\sqrt{|c|}} & (c < 0). \end{cases}$$

4 (1) $u = \tan \frac{x}{2}$ と置換すると. $\cos x = \frac{1-u^2}{1+u^2}$, $\sin x = \frac{2u}{1+u^2}$, $dx = \frac{2du}{1+u^2}$ より,

$$\begin{aligned} \int \frac{1+\sin x}{\sin x(1+\cos x)} dx &= \int \frac{1+\frac{2u}{1+u^2}}{\frac{2u}{1+u^2}(1+\frac{1-u^2}{1+u^2})} \cdot \frac{2}{1+u^2} du = \int \frac{u^2+2u+1}{2u} du = \frac{1}{2} \int \left(u+2+\frac{1}{u}\right) du \\ &= \frac{1}{4}u^2 + u + \frac{1}{2} \log |u| = \left[\frac{1}{4} \tan^2 \frac{x}{2} + \tan \frac{x}{2} + \frac{1}{2} \log \left| \tan \frac{x}{2} \right| \right]. \end{aligned}$$

(2) $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$ より, $\sin^4 x + \cos^4 x = \frac{1}{2} + \frac{1}{2} \cos^2 2x = \frac{3}{4} + \frac{1}{4} \cos 4x$.

よって, $\int (\sin^4 x + \cos^4 x) dx = \left[\frac{3}{4}x + \frac{1}{16} \sin 4x \right]$.

(3) $u = \tan x$ と置換すると, $\cos^2 x = \frac{1}{1+u^2}$, $\sin^2 x = \frac{u^2}{1+u^2}$, $dx = \frac{du}{1+u^2}$ なので,

$$\begin{aligned} \int \frac{dx}{\cos^2 x + 4\sin^2 x} &= \int \frac{1}{\frac{1}{1+u^2} + 4 \cdot \frac{u^2}{1+u^2}} \cdot \frac{du}{1+u^2} = \int \frac{dt}{1+4u^2} = \frac{1}{4} \int \frac{du}{(\frac{1}{2})^2 + u^2} \\ &= \frac{1}{2} \text{Tan}^{-1} 2u = \left[\frac{1}{2} \text{Tan}^{-1}(2 \tan x) \right]. \end{aligned}$$

5 (1) $t = \sqrt{1-x}$ と置換すると, $x = 1-t^2$, $dx = -2t dt$ なので,

$$\begin{aligned} \int_0^1 \frac{dx}{1+x+\sqrt{1-x}} &= \int_1^0 \frac{-2t}{2-t^2+t} dt = - \int_0^1 \frac{2t}{(t-2)(t+1)} dt = -\frac{2}{3} \int_0^1 \left(\frac{2}{t-2} + \frac{1}{t+1} \right) dt \\ &= -\frac{2}{3} \left[2 \log |t-2| + \log |t+1| \right]_0^1 = \left[\frac{2}{3} \log 2 \right]. \end{aligned}$$

(2) $\int_0^{\frac{1}{2}} \frac{\text{Sin}^{-1} x}{\sqrt{1-x^2}} dx = \int_0^{\frac{1}{2}} \text{Sin}^{-1} x (\text{Sin}^{-1} x)' dx = \left[\frac{1}{2} (\text{Sin}^{-1} x)^2 \right]_0^{\frac{1}{2}} = \frac{1}{2} \left(\frac{\pi}{6} \right)^2 = \left[\frac{\pi^2}{72} \right]$.

(3) $u = \tan \frac{x}{2}$ と置換すると, $\sin x = \frac{2u}{1+u^2}$, $dx = \frac{2du}{1+u^2}$ なので,

$$\begin{aligned} \int_0^{\pi/2} \frac{dx}{4+5\sin x} &= \int_0^1 \frac{1}{4+5 \cdot \frac{2u}{1+u^2}} \cdot \frac{2du}{1+u^2} = \int_0^1 \frac{du}{2u^2+5u+2} = \int_0^1 \frac{du}{(2u+1)(u+2)} \\ &= \frac{1}{3} \int_0^1 \left(\frac{2}{2u+1} - \frac{1}{u+2} \right) du = \frac{1}{3} \left[\log \frac{2u+1}{u+2} \right]_0^1 = \left[\frac{1}{3} \log 2 \right]. \end{aligned}$$

6 (1) $0 < x \leq \frac{\pi}{2}$ において, $\frac{2}{\pi}x \leq \sin x \leq x$ なので, $\frac{1}{x^p} \leq \frac{1}{(\sin x)^p} \leq \left(\frac{\pi}{2}\right)^p \frac{1}{x^p}$. $0 < \varepsilon < 1$ に対して

$$\int_{\varepsilon}^{\pi/2} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} \left\{ \left(\frac{\pi}{2}\right)^{1-p} - \varepsilon^{1-p} \right\} & (0 < p \neq 1) \\ \log \frac{\pi}{2\varepsilon} & (p = 1) \end{cases} \quad \text{よって,} \quad \lim_{\varepsilon \rightarrow +0} \int_{\varepsilon}^{\pi/2} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} \left(\frac{\pi}{2}\right)^{1-p} & (0 < p < 1) \\ \infty & (p \geq 1) \end{cases}$$

となる. よって, 広義積分 $\int_0^{\pi/2} \frac{dx}{(\sin x)^p}$ は $0 < p < 1$ のとき収束し, $p \geq 1$ のとき (∞ に) 発散する.

$$(2) (x-a)^2 + y^2 \leq b^2 \Leftrightarrow a - \sqrt{b^2 - y^2} \leq x \leq a + \sqrt{b^2 - y^2} \text{ より,}$$

$$\begin{aligned} \pi^{-1}V &= \int_{-b}^b \{(a + \sqrt{b^2 - y^2})^2 - (a - \sqrt{b^2 - y^2})^2\} dy = \int_{-b}^b 2a \cdot 2\sqrt{b^2 - y^2} dy \\ &= 8a \int_0^b \sqrt{b^2 - y^2} dy = 2\pi ab^2. \quad \therefore V = \boxed{2\pi^2 ab^2}. \end{aligned}$$

(3) L を求めるためには, 第 1 象限の部分の長さを 4 倍すればよい. 第 1 象限の部分は $x = \cos^3 t$, $y = \sin^3 t$ ($0 \leq t \leq \pi/2$) とパラメータ表示でき, $\frac{dx}{dt} = -3\cos^2 t \sin t$, $\frac{dy}{dt} = 3\sin^2 t \cos t$ となるから,

$$L = 4 \int_0^{\pi/2} \sqrt{(-3\cos^2 t \sin t)^2 + (3\sin^2 t \cos t)^2} dt = 12 \int_0^{\pi/2} \cos t \sin t dt = 12 \left[\frac{1}{2} \sin^2 t \right]_0^{\pi/2} = \boxed{6}.$$

レポート問題

- (1) 部分分数分解 $\frac{3x^2 - 2x - 3}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ の分母を払って, $3x^2 - 2x - 3 = A(x^2+1) + (Bx+C)(x+1)$. 係数を比較すると, $3 = A+B$, $-2 = B+C$, $-3 = A+C$ となるので, これを解くと, $A=1, B=2, C=-4$. ($x=-1, i$ を代入して求めてもよい.) よって $\frac{3x^2 - 2x - 3}{(x+1)(x^2+1)} = \frac{1}{x+1} + \frac{2x-4}{x^2+1}$ となる. これを積分すれば

$$\begin{aligned} \int_0^1 \frac{3x^2 - 2x - 3}{(x+1)(x^2+1)} dx &= \int_0^1 \frac{dx}{x+1} + \int_0^1 \frac{2x-4}{x^2+1} dx = [\log|x+1|]_0^1 + \int_0^1 \frac{2x}{x^2+1} dx - 4 \int_0^1 \frac{dx}{x^2+1} \\ &= \log 2 + [\log(x^2+1)]_0^1 - 4 [\tan^{-1}x]_0^1 = \boxed{2\log 2 - \pi} \end{aligned}$$

- (2) $x = \tan \frac{t}{2}$ と置換すると

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \frac{\sin x + 3 \cos x}{1 + \sin x + \cos x} dx &= \int_0^1 \frac{\frac{2t}{1+t^2} + \frac{3(1-t^2)}{1+t^2}}{1 + \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} = \int_0^1 \frac{2t+3-3t^2}{2(1+t)} \cdot \frac{2dt}{1+t^2} \\ &= \int_0^1 \frac{-3t^2+2t+3}{(1+t)(1+t^2)} dt \stackrel{(1)}{=} \boxed{-2\log 2 + \pi} \end{aligned}$$

- (3) 部分積分を行うと

$$\begin{aligned} \int_0^{\frac{1}{\sqrt{2}}} x \cos^{-1} x dx &= \left[\frac{x^2 \cos^{-1} x}{2} \right]_0^{\frac{1}{\sqrt{2}}} + \int_0^{\frac{1}{\sqrt{2}}} \frac{x^2}{2\sqrt{1-x^2}} dx = \frac{\pi}{16} + \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}} \cos t dt \\ &= \frac{\pi}{16} + \frac{1}{2} \int_0^{\frac{\pi}{4}} \sin^2 t dt = \frac{\pi}{16} + \frac{1}{2} \int_0^{\frac{\pi}{4}} \frac{1 - \cos 2t}{2} dt = \frac{\pi}{16} + \frac{1}{4} \left[t - \frac{\sin 2t}{2} \right]_0^{\frac{\pi}{4}} = \boxed{\frac{\pi-1}{8}} \end{aligned}$$

$$(4) \int \frac{dx}{\sqrt{9-4x^2}} = \boxed{\frac{1}{2} \sin^{-1} \frac{2x}{3}}.$$

- (5) $t = \sqrt{x-2}$ と置換すると $x = t^2 + 2$ より $\frac{dx}{dt} = 2t$ だから

$$\int_4^8 \frac{dx}{x\sqrt{x-2}} = \int_{\sqrt{2}}^{\sqrt{6}} \frac{2tdt}{(t^2+2)t} = \int_{\sqrt{2}}^{\sqrt{6}} \frac{2dt}{t^2+2} = \left[\sqrt{2} \tan^{-1} \frac{t}{\sqrt{2}} \right]_{\sqrt{2}}^{\sqrt{6}} = \sqrt{2} \left(\frac{\pi}{3} - \frac{\pi}{4} \right) = \boxed{\frac{\sqrt{2}\pi}{12}}$$